



# Numerical analysis for the heat flux in a mixed elliptic problem to obtain a discrete steady-state two-phase Stefan problem

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**Domingo Alberto TARZIA**

**Février 1992**



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# NUMERICAL ANALYSIS FOR THE HEAT FLUX IN A MIXED ELLIPTIC PROBLEM TO OBTAIN A DISCRETE STEADY-STATE TWO-PHASE STEFAN PROBLEM

## ANALYSE NUMERIQUE DU FLUX DE CHALEUR DANS UN PROBLEME ELLIPTIQUE MIXTE POUR OBTENIR UN CAS STATIONNAIRE DISCRET DU PROBLEME DE STEFAN A DEUX PHASES

Domingo Alberto TARZIA (\*)

### Abstract

We consider a material  $\Omega \subset \mathbb{R}^n$  which occupies a convex polygonal bounded domain, with regular boundary  $\Gamma = \Gamma_1 \cup \Gamma_2$  with  $\text{meas}(\Gamma_1) = |\Gamma_1| > 0$  and  $|\Gamma_2| > 0$ . We assume, without loss of generality, that the melting temperature is  $0^\circ\text{C}$ . We apply a temperature  $b = \text{Const.} > 0$  on  $\Gamma_1$  and a heat flux  $q = \text{Const.} > 0$  on  $\Gamma_2$ . We consider a steady-state heat conduction problem in  $\Omega$ .

We consider a regular triangulation of the domain  $\Omega$  with Lagrange triangles of type 1. We study sufficient (and/or necessary) conditions for the heat flux  $q$  on  $\Gamma_2$  to obtain a change of phase (steady-state two-phase discretized Stefan problem) into the corresponding discretized domain, that is a discrete temperature of non-constant sign in  $\Omega$ .

### Résumé

On considère un matériel  $\Omega \subset \mathbb{R}^n$ , un domaine polygonal borné et convexe avec une frontière régulière  $\Gamma = \Gamma_1 \cup \Gamma_2$  avec  $\text{mes}(\Gamma_1) = |\Gamma_1| > 0$  et  $|\Gamma_2| > 0$ . On suppose, sans perte de généralité, que la température de changement de phase est  $0^\circ\text{C}$ . On considère une température  $b = \text{Const.} > 0$  sur  $\Gamma_1$  et un flux de chaleur  $q = \text{Const.} > 0$  sur  $\Gamma_2$ . On considère le cas stationnaire du problème de conduction de la chaleur dans  $\Omega$ .

On considère une triangulation régulier de  $\Omega$  avec des triangles de Lagrange de type 1. On étudie des conditions suffisantes (et/ou nécessaires) pour le flux de chaleur  $q$  sur  $\Gamma_2$  pour obtenir un changement de phase (cas stationnaire discret d'un problème de Stefan à deux phases) dans le domaine discrète correspondante, c'est-à-dire une température discrète de signe non-constante dans  $\Omega$ .

Key words : Steady-state Stefan problem, finite element method, mixed elliptic problem, numerical analysis, variational inequalities, error bounds.

Mots Clés : Problème de Stefan stationnaire, méthode d'éléments finis, problème elliptique mixte, analyse numérique, inéquations variationnelles, estimation de l'erreur.

AMS Subject Classification : 35R35, 35J85, 65N15, 65N30.

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# I.- INTRODUCTION.

We consider a material  $\Omega \subset \mathbb{R}^n$  which occupies a convex polygonal bounded domain, with a regular boundary  $\Gamma = \Gamma_1 \cup \Gamma_2$  with  $\text{meas}(\Gamma_1) = |\Gamma_1| > 0$  and  $|\Gamma_2| > 0$ . We assume, without loss of generality, that the melting temperature is  $0^\circ\text{C}$ . We apply a temperature  $b = \text{Const.} > 0$  on  $\Gamma_1$  and a heat flux  $q = \text{Const.} > 0$  on  $\Gamma_2$ . We consider a steady-state heat conduction problem in  $\Omega$ . Following [7], we study the temperature  $\theta = \theta(x)$  for  $x \in \Omega$ . The set  $\Omega$  can be written by

$$(1) \quad \Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{L} .$$

where

$$(2) \quad \begin{aligned} \Omega_1 &= \left\{ x \in \Omega / \theta(x) < 0 \right\} \text{ (liquid phase) ,} \\ \Omega_2 &= \left\{ x \in \Omega / \theta(x) > 0 \right\} \text{ (solid phase) ,} \\ \mathcal{L} &= \left\{ x \in \Omega / \theta(x) = 0 \right\} \text{ (free boundary) ,} \end{aligned}$$

are the liquid phase, the solid phase and the free boundary which separates them respectively. The temperature  $\theta$  can be represented in  $\Omega$  in the following way

$$(3) \quad \begin{aligned} \theta_1(x) &< 0 , \quad x \in \Omega_1 , \\ \theta(x) &= 0 , \quad x \in \mathcal{L} , \\ \theta_2(x) &> 0 , \quad x \in \Omega_2 , \end{aligned}$$

and satisfies the following conditions :

$$(4) \quad \begin{aligned} \Delta \theta_i &= 0 \quad \text{in } \Omega_i \quad (i = 1, 2) , \\ \theta_1 = \theta_2 = 0 , \quad k_1 \frac{\partial \theta_1}{\partial n} &= k_2 \frac{\partial \theta_2}{\partial n} \quad \text{on } \mathcal{L} , \\ \theta_2|_{\Gamma_1} &= b > 0 , \\ -k_2 \frac{\partial \theta_2}{\partial n}|_{\Gamma_2} &= q \quad \text{if } \theta|_{\Gamma_2} > 0 , \\ -k_1 \frac{\partial \theta_1}{\partial n}|_{\Gamma_2} &= q \quad \text{if } \theta|_{\Gamma_2} < 0 , \end{aligned}$$

where  $k_i > 0$  is the thermal conductivity in  $\Omega_i$  ( $i=1$  : solid phase,  $i = 2$  : liquid phase). If we introduce the new unknown function [2, 7]

$$(5) \quad u = k_2 \theta^+ - k_1 \theta^- \quad \left( \theta = \frac{1}{k_2} u^+ - \frac{1}{k_1} u^- \right) \quad \text{in } \Omega ,$$

where  $\theta^+$  and  $\theta^-$  represent the positive part and the negative part of the function  $\theta$  respectively, then problem (4) is transformed into the following mixed elliptic problem

$$(6) \quad \begin{aligned} \Delta u &= 0 \quad \text{dans } \Omega , \\ u|_{\Gamma_1} &= B , \\ -\frac{\partial u}{\partial n}|_{\Gamma_2} &= q , \end{aligned}$$

whose variational formulation is given by

$$(7) \quad \begin{aligned} a(u, v-u) &= L(v-u) \quad , \quad \forall v \in K , \\ u &\in K , \end{aligned}$$

where

$$(8) \quad \begin{aligned} V &= H^1(\Omega) \quad , \quad B = k_2 b > 0 \quad \text{on } \Gamma_1 , \\ K &= \left\{ v \in V / v|_{\Gamma_1} = B \right\} , \quad V_0 = \left\{ v \in V / v|_{\Gamma_1} = 0 \right\} , \\ a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v \, dx \quad , \quad L(v) = L_q(v) = - \int_{\Gamma_2} q v \, d\gamma . \end{aligned}$$

Moreover, the solution of (7) is characterized by the following minimum problem [4]

$$(9) \quad \begin{aligned} J(u) &\leq J(v) \quad , \quad \forall v \in K , \\ u &\in K , \end{aligned}$$

where

$$(10) \quad J(v) = J_q(v) = \frac{1}{2} a(v, v) - L_q(v) = \frac{1}{2} a(v, v) + \int_{\Gamma_2} q v \, d\gamma .$$

We can define the real fonction  $f: \mathbb{R}^+ \rightarrow \mathbb{R}$  in the following way

$$(11) \quad f(q) = J_q(u(q)) = \frac{1}{2} a(u(q), u(q)) + q \int_{\Gamma_2} u_q \, d\gamma ,$$

where  $u = u(q)$  is the unique solution of the variational equality (7) for each heat flux  $q > 0$  (for a given  $B > 0$ ).

For the continuous problem (6) or (7), a sufficient condition to have a steady-state two-phase Stefan problem (i.e. the solution  $u(q)$  of (7) is a function of non-constant sign in  $\Omega$ ) was obtained in [8, 9].

**Theorem 1 .—** (i) Function  $f$  is derivable. Moreover,  $f'$  is a continuous and strictly decreasing function, and it is given by the following expression

$$(12) \quad f'(q) = \int_{\Gamma_2} u(q) \, d\gamma .$$

(ii) There exists a geometrical constant  $C = C(\Omega, \Gamma_1, \Gamma_2) > 0$  such that

$$(13) \quad a(u(q), u(q)) = C q^2 , \quad f(q) = -\frac{C}{2} q^2 + B |\Gamma_2| q , \quad \forall q > 0 .$$

Moreover, the constant  $C > 0$  is given by

$$(14) \quad C = a(u_3, u_3) = \int_{\Gamma_2} u_3 \, d\gamma > 0 ,$$

where  $u_3 \in V_0$  ( $u(q) = B - q u_3$  in  $\Omega$ ) is the unique solution of the following mixed elliptic problem

$$(15) \quad \begin{aligned} \Delta u_3 &= 0 \text{ in } \Omega , \\ u_3|_{\Gamma_1} &= 0 , \quad \frac{\partial u_3}{\partial n}|_{\Gamma_2} = 1 , \end{aligned}$$

whose variational formulation is given by

$$(16) \quad \begin{aligned} a(u_3, v) &= \int_{\Gamma_2} v \, d\gamma , \quad \forall v \in V_0 , \\ u_3 &\in V_0 . \end{aligned}$$

(iii) If

$$(17) \quad q > q_0(B)$$

then (6) or (7) represents a steady-state two-phase Stefan problem (i.e. the solution  $u(q)$  of problem (7) is a function of non-constant sign in  $\Omega$ ), where  $q_0 = q_0(B)$  is given by

$$(18) \quad q_0(B) = \frac{B |\Gamma_2|}{C} , \quad \forall B > 0 .$$

(iv) If the function  $u(q)$  is constant over  $\Gamma_2$ , then the sufficient condition (given by (17)) is also necessary.

Proof .— See [8, 9].

Now, we consider  $\tau_h$ , a regular triangulation of the polygonal domain  $\Omega$  with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class  $C^0$ , where  $h > 0$  is a parameter which goes to zero. We can take  $h$  equals to the longest side of the triangles  $T \in \tau_h$  and we can approximate  $V_0$  by [1] :

$$(19) \quad V_h = \left\{ v_h \in C^0(\bar{\Omega}) / v_h|_T \in P_1(T), \forall T \in \tau_h, v_h|_{\Gamma_1} = 0 \right\},$$

where  $P_1$  is the set of the polynomials of degree less or equals than 1. Let  $\pi_h$  be the corresponding linear interpolation operator. Then, we can consider that there exists a constant  $C_0 > 0$  (independent of the parameter  $h$ ) such that

$$(20) \quad \|v - \pi_h v\|_V \leq C_0 h^{r-1} \|v\|_{r,\Omega}, \quad \forall v \in H^r(\Omega), \text{ with } 1 < r \leq 2.$$

We consider the following finite dimensional approximate variational problem, corresponding to the continuous variational problem (7), given by :

$$(21) \quad \begin{aligned} a(u_h, v_h) &= L(v_h), \quad \forall v_h \in V_h, \\ u_h &\in K_h = B + V_h, \end{aligned}$$

and we can obtain the following results.

Lema 2 .— We have

$$(22) \quad \lim_{h \rightarrow 0^+} \|u_h - u\|_V = 0,$$

where  $u$  is the unique solution of the variational equality (7).

Proof .— Owing to  $\text{meas}(\Gamma_1) > 0$ , we have that the bilinear form  $a$  is coercitivity over  $V_0$ , that is [4] :

$$(23) \quad \exists \alpha > 0 / a(v, v) = \|v\|_{V_0}^2 \geq \alpha \|v\|_V^2, \quad \forall v \in V_0,$$

and therefore  $\|\cdot\|_{V_0}$  and  $\|\cdot\|_V$  are two equivalent norms in  $V_0$ . We follow a similar method

developped in [1].

Corollary 3 .– If we define

$$(24) \quad \theta_h = \frac{1}{k_2} u_h^+ - \frac{1}{k_1} u_h^- \in V, \quad \theta = \frac{1}{k_2} u^+ - \frac{1}{k_1} u^- \in V$$

then we have

$$(25) \quad \lim_{h \rightarrow 0^+} \|\theta_h - \theta\|_H = 0,$$

where  $H = L^2(\Omega)$ .

Proof .– If we consider the scalar product in  $H$ , defined by

$$(26) \quad (u, v) = \int_{\Omega} u v \, dx,$$

then, we deduce

$$(27) \quad \begin{aligned} \|u_h - u\|_H^2 &= \|u_h^+ - u^+\|_H^2 + \|u_h^- - u^-\|_H^2 + 2(u_h^+, u^-) + \\ &+ 2(u_h^+, u^+) \geq \|u_h^+ - u^+\|_H^2 + \|u_h^- - u^-\|_H^2, \end{aligned}$$

that is

$$(28) \quad \text{Max} (\|u_h^+ - u^+\|, \|u_h^- - u^-\|) \leq \|u_h - u\|_H.$$

From (24) we obtain :

$$(29) \quad \begin{aligned} \|\theta_h - \theta\|_H &\leq \frac{1}{k_2} \|u_h^+ - u^+\|_H + \frac{1}{k_1} \|u_h^- - u^-\|_H \leq \\ &\leq \left( \frac{1}{k_1} + \frac{1}{k_2} \right) \|u_h - u\|_H, \end{aligned}$$

i.e. (25).

The goal of this paper is to consider the numerical analysis of the inequality (17). We study sufficient (and/or necessary) conditions for the constant heat flux  $q$  on  $\Gamma_2$  to obtain a change of phase (steady-state two-phase discretized Stefan problem) into the corresponding discretized domain, that is a discrete temperature of non-constant sign in  $\Omega$ . We obtain that :

i) there exists a constant  $C_h > 0$  (which depends only of the geometry of the domain  $\Omega$  for each  $h > 0$  and it is characterized by a variational problem) such that if  $q > q_{0h}(B) = B|\Gamma_2|/C_h$  then the steady-state discretized problem presents two phases.



ii) we have the estimations  $C_h < C$  and  $q_0(B) < q_{0h}(B)$  where  $C$  and  $q_0(B)$  have been obtained for the continuous problem by (14) and (18) respectively [9].

iii) we deduce an error bounds for  $C - C_h$  and  $q_{0h}(B) - q_0(B)$  as a function of the parameter  $h$ .

In other words, we obtain for the mixed elliptic discretized problem, defined by  $u_h$ , analogous conditions to the ones obtained for the corresponding continuous problem [9], defined by  $u$ .

## II. — INEQUALITY FOR THE HEAT FLUX IN THE DISCRETIZED PROBLEM.

For each  $q > 0$  we consider the functions  $u(q) \in K$  and  $u_h(q) \in K_h$ , as the unique solution of the variational equalities (7) (continuous problem) and (21) (discrete problem) respectively. We define the real function  $f_h : \mathbb{R}^+ \rightarrow \mathbb{R}$ , for each  $h > 0$ , in the following way

$$(30) \quad f_h(q) = J_q(u_h(q)) = \frac{1}{2} a(u_h(q), u_h(q)) + q \int_{\Gamma_2} u_h(q) d\gamma, \quad q > 0.$$

Therefore, we obtain the following properties :

Theorem 4 .— (i) If  $u_i = u_h(q_i)$  is the solution of (21) for  $q_i > 0$  ( $i = 1, 2$ ), then we have the following equalities :

$$(31) \quad a(u_2 - u_1, u_2 - u_1) = (q_1 - q_2) \int_{\Gamma_2} (u_2 - u_1) d\gamma,$$

$$(32) \quad a(u_2, u_2) - a(u_1, u_1) = a(u_2 + u_1, u_2 - u_1) = (q_1 + q_2) \int_{\Gamma_2} (u_1 - u_2) d\gamma.$$

(ii) For all real numbers  $q > 0$  and  $\Delta$  such that  $(q + \Delta) > 0$ , we obtain the following estimations :

$$(33) \quad \left\| \frac{1}{\Delta} [u_h(q) - u_h(q + \Delta)] \right\|_V \leq D_1 = \frac{\|\gamma_0\|}{\alpha} |\Gamma_2|^{1/2},$$

$$(34) \quad \left\| \frac{1}{\Delta} [u_h(q) - u_h(q + \Delta)] \right\|_{L^2(\Gamma_2)} \leq D_2 = D_1 \|\gamma_0\|,$$

where  $\gamma_0$  is the linear and continuous trace operator, defined over  $V$ . Moreover, the function

$$(35) \quad q > 0 \rightarrow \int_{\Gamma_2} u_h(q) d\gamma \in \mathbb{R},$$

is a continuous and strictly decreasing function.

(iii) Function  $f_h = f_h(q)$  is derivable. Moreover,  $f_h'$  is a continuous and strictly decreasing function and given by the following expression

$$(36) \quad f_h'(q) = \int_{\Gamma_2} u_h(q) \, d\gamma ,$$

Proof .— (i) If we take  $v = u_2 - u_1 \in V_h$  in the variational equality corresponding to  $u_1$  and  $v = u_1 - u_2 \in V_h$  in the one corresponding to  $u_2$ , and we add up and subtract both equalities, then we obtain (31) and (32) respectively.

(ii) Taking into account (23), the Cauchy-Schwarz inequality and the continuity of the operator  $\gamma_0$  we deduce (33). From (33) and the continuity of  $\gamma_0$  we have (34). Therefore we have (35) because

$$(37) \quad \left| \int_{\Gamma_2} [u_h(q) - u_h(q + \Delta)] \, d\gamma \right| \leq D_2 |\Gamma_2|^{1/2} \Delta .$$

Moreover, the monotony property is a consequence of (31).

(iii) From (30) and elementary computations, we deduce

$$(38) \quad \frac{1}{\Delta} [f_h(q + \Delta) - f_h(q)] = \frac{1}{2} \int_{\Gamma_2} [u_h(q) + u_h(q + \Delta)] \, d\gamma ,$$

that is (36), by using (35).

Lema 5 .— We have the following expressions :

$$(39) \quad a(u_h(q), u_h(q)) = q B |\Gamma_2| - q \int_{\Gamma_2} u_h(q) \, d\gamma ,$$

$$(40) \quad f_h(q) = q B |\Gamma_2| - \frac{1}{2} a(u_h(q), u_h(q)) ,$$

$$(41) \quad f_h(q) = \frac{q B |\Gamma_2|}{2} + \frac{q}{2} \int_{\Gamma_2} [u_h(q) \, d\gamma ,$$

$$(42) \quad q f_h''(q) = f_h'(q) - B |\Gamma_2| ,$$

$$(43) \quad \frac{d}{dq} [a(u_h(q), u_h(q))] = \frac{2}{q} a(u_h(q), u_h(q)) = 2 B |\Gamma_2| - 2 f_h'(q) .$$

Proof .— Taking  $v_h = B - u_h(q) \in V_h$  in (21) we obtain (31). We deduce (40) and (41) by using (30) and (39). From (36), (39)–(41) we obtain (42) and (43) by derivating with respect to variable  $q$ .

Theorem 6 .– (i) There exists a constant  $C_h > 0$  such that

$$(44) \quad f_h(q) = q B |\Gamma_2| - \frac{1}{2} C_h q^2, \quad \forall q > 0,$$

$$(45) \quad a(u_h(q), u_h(q)) = C_h q^2, \quad \forall q > 0.$$

(ii) If  $q > q_{0h}(B)$ , then problem (21) represents a discretized steady-state two-phase Stefan problem (i.e.  $u_h(q)$  is a function of non-constant sign in  $\Omega$ ), where

$$(46) \quad q_{0h}(B) = \frac{B |\Gamma_2|}{C_h}.$$

(iii) The constant  $C_h > 0$  can be computed by the expression

$$(47) \quad C_h = \frac{1}{q} \int_{\Gamma_2} [B - u_h(q)] d\gamma,$$

for any  $q > 0$ . Moreover,  $C_h$  is given by

$$(48) \quad C_h = a(u_{3h}, u_{3h}) = \int_{\Gamma_2} u_{3h} d\gamma,$$

where  $u_{3h}$  is the unique solution of the variational equality

$$(49) \quad \begin{aligned} a(u_{3h}, v_h) &= \int_{\Gamma_2} v_h d\gamma, \quad \forall v_h \in V_h, \\ u_{3h} &\in V_h. \end{aligned}$$

Proof .– (i) Function  $f_h = f_h(q)$  satisfies  $f_h'''(q) = 0$ ,  $\forall q > 0$  because (42). Owing to

$$(50) \quad u_h(0^+) = B, \quad f_h(0^+) = B |\Gamma_2| q, \quad f_h'(0^+) = B |\Gamma_2|,$$

there exists some constant  $C_h > 0$  such that (44) and (45) are holds.

(ii) It follows taking into account

$$(51) \quad f_h'(q_{0h}(B)) = 0,$$

and the monotony property of function  $f_h'$ .

(iii) From (39) and (45) we obtain (47) for any  $q > 0$ . Moreover, if we define  $w_h = w_h(q) \in V_h$  by the expression

$$(52) \quad w_h(q) = \frac{B - u_h(q)}{q},$$

then, we have :

$$(53) \quad a(w_h(q), v_h) = \frac{1}{q} a(B - u_h(q), v_h) = -\frac{1}{q} a(u_h(q), v_h) = \int_{\Gamma_2} v_h d\gamma, \quad \forall v_h \in V_h,$$

that is  $w_h(q) = u_{3h}$  by uniqueness of the solution of the variational equality (49). then, we deduce (48).

Theorem 7 .- (i) We have the following equality :

$$(54) \quad a(u(q), u_h(q)) = C_h q^2, \quad \forall q > 0.$$

(ii) We have the following inequalities :

$$(55) \quad (a) C_h < C, \quad (b) q_0(B) < q_{0h}(B).$$

Proof .- (i) If we take  $v = u_h(q) \in K_h = B + V_h \subset B + V_0 = K$  in the variational equality (7), and we take into account the expressions (13) and (44), then we obtain (54).

(ii) On the other hand, from (23) and (54) we have

$$(56) \quad \alpha \|u(q) - u_h(q)\|_V^2 \leq a(u(q) - u_h(q), u(q) - u_h(q)) = (C - C_h) q^2,$$

that is (55a). Moreover, (55b) follows from (18), (46) and (55).

Now, we shall use the interpolation result (20) for the function  $u_3 \in H^r(\Omega)$ , as a hipotesis of regularity of the continuous problem (7) (in general,  $1 < r < \frac{3}{2}$  [3, 5, 6] ). In [8], we present three examples with explicit solution. In these cases, we have  $u(q), u_3 \in C^\infty(\Omega)$ .

Theorem 8 .- We have the following relations and estimations :

$$(57) \quad a(u(q) - u_h(q), v_h) = 0, \quad \forall v_h \in V_h,$$

$$(58) \quad (C - C_h) q^2 = a(u(q) - u_h(q), u(q) - u_h(q)) \leq \inf_{v_h \in V_h} a(u(q) - v_h, u(q) - v_h),$$

$$(59) \quad 0 < C - C_h \leq C_0^2 h^{2(r-1)} \|u_3\|_{r,\Omega}^2,$$

$$(60) \quad 0 < q_{0_h}(B) - q_0(B) \leq \frac{C_0^2 h^{2(r-1)}}{C} \|u_3\|_{r,\Omega}^2 q_{0_h}(B).$$

Proof .— If we take  $v = v_h \in V_h \subset V_0$  in the variational equality (7) and we subtract it with the variational equality (21), we obtain (57). By using (54), (56) and (57) we deduce

$$\begin{aligned} (61) \quad & a(u(q) - u_h(q), u(q) - u_h(q)) = a(u(q) - u_h(q), u(q)) - \\ & a(u(q) - u_h(q), u_h(q)) = a(u(q) - u_h(q), u(q)) - a(u(q) - u_h(q), v_h) = \\ & = a(u(q) - u_h(q), u(q) - v_h) \leq [a(u(q) - u_h(q), u(q) - u_h(q))]^{\frac{1}{2}} \cdot \\ & \cdot [a(u(q) - v_h, u(q) - v_h)]^{\frac{1}{2}}, \quad \forall v_h \in V_h, \end{aligned}$$

because  $a(.,.)$  is a scalar product in  $V_0$ , then we obtain (58).

By using (58), the facts that

$$(62) \quad \Pi_h(u(q)) \in B + V_h, \quad u(q) - \Pi_h(u(q)) \in V_h$$

and the interpolation result (20), we deduce (59). The relation (60) is obtained by using the definition of  $q_{0_h}(B)$  and  $q_0(B)$ , and (59).

Remark 1 .— If we only have  $u(q) \in V$  (i.e.  $u_3 \in V$ ) we can obtain

$$(63) \quad 0 < C - C_h \leq \frac{1}{q^2} \|u(q) - \Pi_h(u(q))\|_V^2 = \|u_3 - \Pi_h(u_3)\|_V^2,$$

where the second term converges to zero when  $h \rightarrow 0^+$  [1], but we cannot give an order of convergence.

Remark 2 .— If the constant heat flux on  $\Gamma_2$  verifies the inequality  $q > q_{0_h}(B)$ , then both discrete and continuous problem represent a steady-state two-phase Stefan problem, that is, their temperatures are of non-constant sign in  $\Omega$ .

Remark 3 .— When the function  $u_h(q)$  is constant on  $\Gamma_2$  (as a function of  $x \in \Gamma_2$ ), then the sufficient condition, given by the Theorem 6—part (ii), is also a necessary condition to have a

two-phase discrete problem, because

$$(64) \quad \int_{\Gamma_2} u_h(q) \, d\gamma < 0 \Leftrightarrow u_h(q) < 0 \text{ on } \Gamma_2 .$$

Theorem 9 .— If we consider  $h, B > 0$ , and  $0 < \epsilon_0 < 1$  ( $\epsilon_0$  is a parameter to be chosen arbitrarily), then we have the following estimations :

$$(65) \quad q_0(B) < q_{0_h}(B) \leq \frac{q_0(B)}{\epsilon_0} \quad \text{and} \quad C_h \geq C \epsilon_0, \quad \forall h \leq h_r(\epsilon_0),$$

$$(66) \quad 0 < q_{0_h}(B) - q_0(B) \leq \frac{C_0^2 |u_3|_{r,\Omega}^2}{C \epsilon_0} q_0(B) h^{2(r-1)}, \quad \forall h \leq h_r(\epsilon_0),$$

where

$$(67) \quad h_r(\epsilon_0) = \left( \frac{C(1 - \epsilon_0)}{C_0^2 |u_3|_{r,\Omega}^2} \right)^{\frac{1}{2(r-1)}} .$$

Proof .— From (60) we deduce

$$(68) \quad A(h) q_{0_h}(B) \leq q_0(B),$$

where

$$(69) \quad A(h) = 1 - \frac{C_0^2 |u_3|_{r,\Omega}^2}{C} h^{2(r-1)} < 1 .$$

If we consider, for each parameter  $0 < \epsilon_0 < 1$ , the following equivalence :

$$(70) \quad 0 < \epsilon_0 < A(h) < 1 \Leftrightarrow 0 < h < h_r(\epsilon_0),$$

we can deduce the inequalities (65) and (66).

Corollary 10 .— If  $B > 0$ , then we have the following limit

$$(71) \quad \lim_{h \rightarrow 0^+} q_{0_h}(B) = q_0(B) .$$

Remark 4 .— If  $r = 2$  then the convergence in Corollary 10 is of the order of  $h$ .

Remark 5 .— Every thing we proved in this paper is still valid if the boundary  $\Gamma$  of the bounded domain  $\Omega$  is represented by the union of the portions ( $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ ) such that they have the following characteristics :

- (i)  $\Gamma_1$  and  $\Gamma_2$  have the same conditions as the ones previously described in (4).
- (ii)  $\Gamma_3$  is a wall impermeable to heat, i.e. we have  $\frac{\partial \theta}{\partial n} \big|_{\Gamma_3} = 0$  in (4) and therefore  $\frac{\partial u}{\partial n} \big|_{\Gamma_3} = 0$  in (6).

Moreover, the first example considered in [8] verifies this condition.

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